Using Classroom Episodes to Foster Prospective Teachers' Didactical Knowledge: Issues for Teacher Education[1]

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After briefly analysing a classroom episode, we discuss aspects of the teacher's didactical knowledge, namely in its mathematical and instructional dimensions, as reflected in the answers of three prospective teachers to a written assignment based on the episode. We then raise some issues regarding initial teacher education, anchored in the notion of didactical knowledge.

Classroom Episodes and Teachers' Didactical Knowledge

Portugal is currently implementing a new mathematics curriculum for basic education (grades 1 to 9, pupils aged 6 to 14) (ME, 2007). This curriculum introduces some changes in the set of mathematical topics to be taught at those grade levels, and stresses three *transversal skills* – problem solving, reasoning, and communication – assigning them an important status concerning the overarching learning objectives. However, in our opinion, the major reform lies in the orientations given for the dynamics of the mathematics classroom, as there is a strong emphasis on the teacher's and the students' active roles inside a community of learners.

The typical structure of teacher education programs has also changed, following European guidelines, as a result of the Bologna process. We draw our attention to the preparation of mathematics teachers for grades 7 through 12 (pupils aged 12 to 17). Currently, prospective teachers enrol in a master's degree after having earned a bachelor one in a scientific area with a minimum of 120 credit hours in mathematics courses. It is only during the master's degree that they take mathematics education courses and realize a school practicum. However, teacher education programs can differ significantly from institution to institution, namely in terms of courses offered and the student teaching experience. This general structure of teacher preparation has been in place for a few years.

As mathematics educators, we pay special attention to the work around issues of classroom communication and to the teacher's role in that process, both in pre-service and in in-service teacher education contexts (Bishop & Goffree, 1986; Tomás Ferreira, 2005; Martinho & Ponte, 2009; Ruthven, Hofmann, & Mercer, 2011). In fact, if pupils learn to communicate by communicating, teachers also learn managing classroom communication by experiencing and reflecting upon situations that involve various challenges regarding that dimension of their role in the classroom. Analysing and discussing short classroom episodes is an important way to achieve that goal (Bishop & Goffree, 1986, Ruthven et al., 2011; Tomás Ferreira, Martinho, & Menezes, 2011).

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Teachers' didactical knowledge is fundamental for engaging students in meaningful mathematical activity. The notion of didactical knowledge is not consensual, and several authors have elaborated on this idea (Ponte, 1999, in press). In our study, we follow Ponte's (1999) perspective – a teacher's didactical knowledge relates to aspects of practice, being a kind of knowledge "essentially oriented towards the action" (p. 61), and encompassing four dimensions: knowledge of curriculum, knowledge of mathematics, knowledge of students and their learning processes, and knowledge of classroom instructional processes (Ponte & Oliveira, 2002). A teacher's didactical knowledge has a dynamic character, as the experiences and situations of practice the teacher encounters in the classroom contribute to its development and constant reformulation.

Acknowledging the inter-related nature of all dimensions of the didactical knowledge, we focus here on mathematical knowledge and on instructional knowledge (Ponte, in press). In this paper, we briefly analyse one classroom episode – *Rita and Prime Numbers* (Boavida, 2001, adapted from Prince, 1998) – focusing our attention on the teacher's actions and discussing some aspects of the teacher's didactical knowledge that support those actions. We present three prospective teachers' analyses of the episode and discuss aspects of their didactical knowledge that emerge in those analyses. Our main goal is to problematize aspects of the recent general structure and organization of initial teacher education programs, based on the notion of didactical knowledge.

"Rita and Prime Numbers": A Classroom Episode

The episode *Rita and Prime Numbers*, presented in Figure 1, happens in a classroom corresponding to Portuguese 7th graders (pupils aged 12). The teacher proposed a closed task (to list all prime numbers up to 50) with a low level of cognitive demand. However, building on a student's comment, the teacher's actions quickly rose the cognitive demand, transforming the task from a simple exercise to a task engaging students in higher order thinking processes, such as proving and discussing aspects of elementary logics (such as implications, reciprocals, examples, and counter-examples). Specifically, what did the teacher do? Which instructional actions characterize the course of events in this episode? We now turn to discussing some of the teacher's main decisions in the episode.

A first decision of the teacher was not to immediately validate Rita's idea. Instead, the teacher left the validation process to the students, hoping to have them presenting their finding to the entire class. In fact, a few moments after sharing her finding with the teacher, Rita went to the board and wrote her answer to the initial task; only after did she communicate her finding to her classmates: "The prime numbers, except 2 and 5, end in 1, 3, 7, or 9". The teacher made a second decision that affected the direction of the discourse: she extended the initial task by building on Rita's finding, challenging the students to check if Rita's idea would work for other prime numbers. The students' reaction was not surprising: they randomly chose several numbers and tested Rita's finding for those numbers. Since they were not able to find any number that could contradict Rita's idea, the class naturally accepted its truthfulness.

Being aware that the proof of a conjecture, and the role of examples in that process, were at stake, the teacher made a third decision which showed to be crucial in the unfolding of events: she wrote on the board "Rita's conjecture: All prime numbers, except 2 and 5, end in 1, 3, 7, or 9". The writing of the word conjecture was not naïve. Indeed, the teacher knew that this term could be unfamiliar to some students. She rewrote Rita's conjecture in a way that made it more explicit in its scope, since the episode provides no information about whether Rita had analysed any prime number larger than 50.

Rita's teacher asked her class to find all prime numbers up to 50. After some time, Rita noticed that the prime numbers larger than 5 she had identified so far ended in 1, 3, 7, or 9. She called her teacher to show her this finding. The teacher asked Rita to work with her partner in order to find the best way to share her finding to the class during the collective discussion of the work. Rita wrote on the board all prime numbers smaller than 50 and she read what she had written in her notebook:

> Rita: The prime numbers except 2 and 5 end in 1, 3, 7, or 9.

The teacher then asked the class to analyse if the same thing happened with other prime numbers. The students started checking several cases of prime numbers, some of which much larger than 100, and they did not find anyone that would not end in 1, 3, 7, or 9. Shortly, they were strongly convinced that what Rita had found was true for all prime numbers, regardless of having been checked, because all prime numbers that they would check always ended in one of those digits. At this time, the teacher wrote on the board:

> Rita's conjecture: All prime numbers, except 2 and 5, end in 1, 3, 7 or 9.

She makes sure the students remember the meaning of conjecture and she challenges them to find a process that allows them to be sure if the conjecture is, indeed, valid for all prime numbers and why that is so. The students try to respond to the challenge and, in this process, they reinforce their conviction that the conjecture is true; yet, their work does not progress. The teacher, then, choosing to work with the whole class, decides to write on the board the numbers from 0 to 9, circling 1, 3, 7, and 9. Almost immediately the students offer several suggestions:

> Maria: Teacher, cross out numbers 0 and 5. A prime number larger than 5 cannot end in 0 or 5.
> Teacher: Why?
> Maria: If it ends in 0 or 5, it is a multiple of 5 and, therefore, it is not prime.
> Daniel: You also have to cross out 2, 4, 6, and 8. If it is larger than 2 and is a prime number, it cannot be even!
> Bernardo: Of course not. 2 divides...
> Teacher: So what?
> Bernardo: A prime number can only have two factors.
> Rosa: Yeah. If it ends in 2, 4, 6, or 8, that is because it is even and even numbers are multiples of 2.
> Rita: We only have 1, 3, 7, and 9 left. Therefore, all prime numbers except 2 and 5 end in the way I found. We're now sure of it.
> Inês: Bu the opposite is not true. For example, 21 ends in 1 and it is not prime.
> Teacher: Why isn't 21 prime?
> Several students: Because 3 divides 21; 3 times 7 is 21.
> Bernardo: It has divisors that are different from 1 and 21.
> Teacher: So, check if this is true, or not, this that I am going to write on the board: all numbers ending in 1, 3, 7, or 9 are prime.

One can hear several voices saying "it's not true". They mention 21, 27, 33... (…)

Figure 1. A classroom episode.

The teacher's fourth important decision was to open a space for discussing with her students the meaning of conjecture, anchoring the discussion on Rita's finding and engaging the students in proving or disproving a conjecture. This process explicitly involved the students in justifying their own reasoning. Such a challenge was nothing but easy to them. In fact, they did not go further than reinforcing their own ideas by finding more and more examples which, nonetheless, proved nothing. And the teacher made another important decision: to write on the board all ten digits, circling those which corresponded to the unit's digit of a prime number, following Rita's conjecture (i.e., leaving 1, 3, 7, and 9 circled). Based on the episode, the students seemed to have correctly interpreted the teacher's intention when writing all ten digits: they signalled all possible ways of ending a natural number. Thus, using their knowledge of divisibility criteria, the students quickly started an elimination process of those digits. Using questions such as "Why?" and "So what?", the teacher ensured that the students justified their assertions. Inês also made a discovery: "But the opposite is not true. For example, 21 ends in 1 and is not prime". After making sure that the students understood why 21 was not a prime number, the teacher made another crucial decision: to write on the board the reciprocal of Rita's conjecture, giving an active voice to Inês's finding. And the counter-examples started emerging, easily convincing the students that this new conjecture was not true.

In sum, throughout the episode, the teacher's actions were determinant for the quality of the classroom discourse, raising the cognitive level of the initial task and engaging the students in significant mathematical activity. We must bear in mind that such activity around issues of elementary logics, quite challenging for the students, had not been planned ahead of time. Instead, the teacher took advantage of a teachable moment, realizing the potential of Rita's comment and building on that opportunity to explore with the students some mathematical ideas that were much more demanding than the notion of prime number.

We can identify a few aspects of the teacher's didactical knowledge in the episode *Rita and Prime Numbers*. For example, the teacher listened to the students' contributions to the classroom discourse in a responsive manner (Empson & Jacobs, 2008), valuing all of them as worthy discussing with the whole class, regardless of their correctness or rigorousness in language. The teacher also gave the students the responsibility for proving or refuting both conjectures, orchestrating a productive whole-class discussion (Stein, Engle, Smith & Hughes, 2008) so that there would be a common understanding of conjecture and no answer would be left unjustified. The teacher's actions along the episode were certainly anchored in her mathematical knowledge, which allowed her to recognize a teachable moment triggered by Rita's finding and to build instruction upon it, pushing her students to do mathematics. This is clear in the transformation of the initial task – a task of procedures without connections (Stein & Smith, 1998) – into a task involving processes of proof. Such a new task, given its higher cognitive demand and the fruitful discussion around it, pushed the students to engage in serious mathematical activity.

Aspects of Didactical Knowledge in the Analysis of a Classroom Episode

We have already stressed the importance of discussing, in teacher education contexts, aspects of the teacher's role in managing mathematical communication in the classroom. This is reflected in our own practice as mathematics educators in which the analysis of classroom episodes such as *Rita and Prime Numbers* is a regular activity. The first author is a mathematics educator in a large urban university in northern Portugal. This institution offers a 2-year master's degree which certifies mathematics teachers for grades 7 to 12. The second semester of coursework, immediately before a year-long student teaching experience along with a few courses, encompasses a mathematics education course that addresses issues of classroom dynamics, including the teacher's role in the general management of classroom meaningful discourse as well as the students' participation in that discourse (NCTM, 1991), and the teacher's challenges in orchestrating productive mathematical discussions (Stein et al., 2008) and in their questioning, listening, and responding approaches (Tomás Ferreira, 2005). With such an agenda, the analysis and discussion of classroom episodes is at the core of the prospective teachers' activities. After completing several assignments throughout the semester, varying in nature, degree of complexity, and organization of students' work (individually, in pairs, in small groups), prospective teachers are typically asked to complete a short written, individual, in-class, and final assignment. In the school year of 2010/11, a cohort of 12 prospective teachers was asked to analyse the episode *Rita and Prime Numbers* in their final assignment. Several questions guided this analysis. We focus here on two of those questions: (1) How do you think the teacher should lead the classroom discourse after the last interventions of the students? and (2) Do you believe Rita's conjecture is proved? If so, why? If not, why? The answers varied significantly both in terms of the mathematical issues underneath the episode and in terms of didactical choices to continue the episode. Next, we briefly analyse the answers of three prospective teachers enrolled in program mentioned above. Since the data gathered for this text is written in Portuguese, instead of illustrating our analyses with excerpts of the prospective teachers' actual work, we resort to our translations of that work.

Júlio completed a bachelor degree in mathematics in the same institution where he is now seeking teacher certification. His suggestion to continue the episode *Rita and Prime Numbers* evidences that he realized that there were two implications involved in the episode, and that one was the reciprocal of the other. Furthermore, he stresses the importance of identifying and distinguishing reciprocal implications and of comprehending the role of examples and counter-examples in the proof and refutation of assertions:

> Based on the students' answer, the teacher should tell them that they had shown the assertion was false, through a whole-class discussion, making them understand that it is enough to give an example that does not verify the assertion for this to be invalid. Then, she should ask the students to relate Rita's conjecture with the latter one, questioning them about their difference[s] and truthfulness, in order to conclude the task.

Júlio shows sensitivity towards an important issue of the teaching and learning process: the development of mathematical reasoning, in particular formulating, testing, and proving (or disproving) conjectures. This prospective teacher has no doubts about the truthfulness of Rita's conjecture: "the conjecture is proved, since the students know that all the numbers end on some digit between 0 and 9, and using divisibility criteria, they managed to exclude the even digits and the 5, remaining 1, 3, 7, and 9". He stresses the role of the class discussion and the importance of using students' (prior) knowledge to help them proving Rita's conjecture: "In this way, and using their own knowledge, the students proved Rita's conjecture, through discussion and exchange of ideas".

Júlio's answers to the written assignment suggest that he recognized the mathematical situation underneath the episode, particularly the existence of reciprocal implications and how they might be proved or disproved. He also gave instructional suggestions that are aligned with current recommendations for mathematics teaching (ME, 2007; NCTM, 2000), especially those related to whole-class discussions as means towards the institutionalization of new knowledge (Stein et al., 2008). In his analysis of the episode, we may say that he shows an adequate mathematical and instructional knowledge, in Ponte's (1999) sense.

Carlos has a similar academic background to Júlio's. His analysis of the episode reveals several problems, which may be anchored in an incorrect interpretation of the episode or in a deficient didactical knowledge. Carlos does not seem to understand that, at the end of the episode, the implication being discussed is the reciprocal of Rita's conjecture:

> After the students said that it was not true, that all prime numbers end in 1, 3, 7, or 9, the teacher should ask them for explanations. Some [students] mention examples that do not verify the conjecture; yet, the teacher should ask for more examples and have them discussing the reason why they are not prime [numbers]. Afterwards, [the teacher] could build on the fact that 9 is not prime since the conjecture said that all numbers ending in 9 were prime.

Carlos's answer suggests that he pays more attention to having students justifying why a given number is prime or composite, than to having students understanding the meaning of a conjecture, the relationship between reciprocal implications (at the level of 7th graders), and the role of examples and counter-examples in the proof or refutation of conjectures. It is not clear why Carlos suggests the teacher to treat number 9 in a *special* way. He gives no importance to summarizing the ideas emerged from the whole-class discussion, despite being a crucial step to assign meaning to those ideas (Stein et al., 2008).

Rita's conjecture seems to have been interpreted in two ways. On one hand, Carlos refers that it may be read as "all prime numbers are all [the numbers] that end in 1, 3, 7, and 9, except 2 and 5"; on the other hand, Rita's conjecture may be read as "prime numbers except 2 and 5 end in 1, 3, 7, or 9". This latter interpretation corresponds to what Rita actually says; thus, Carlos's first interpretation is not grounded in the episode. Nonetheless, it is that first interpretation that Carlos considers as having been discussed with the class: "in this lesson, the only thing that was proved was that the first interpretation is not valid". Indeed, Carlos does not believe that Rita's conjecture was proved during the lesson:

> The way Rita phrased the conjecture seems to indicate that all prime numbers are all odd numbers except those that end in 5. During the lesson, it became clear that this is not true since 21, 27, 33 are odd numbers ending in 1, 7, and 3, and they are not prime. Rita's conjecture was not proved because prime numbers except 2 and 5 end in 1, 3, 7, 9. What was proved was that the numbers ending in 1, 3, 7, and 9 are not always prime.

As we have seen, Carlos does not correctly interpret Rita's conjecture. In addition, he does not realize that the examples provided by the students (21, 27, and 33) are, indeed, examples that destroy the reciprocal of Rita's conjecture, not the conjecture itself. Thus, Carlos does not see those examples as counter-examples for the reciprocal of Rita's conjecture. Furthermore, it seems that he does not recognize a proof of Rita's conjecture in the teacher's and students' joint work. Unlike Júlio, Carlos shows gaps in his didactical knowledge, both at the mathematical and the instructional level (Ponte, 1999). In his analysis of the episode, he does not give importance to the mathematical discussions that occurred nor to the synthesis of the main ideas that were discussed, two aspects that are quite valued in current orientations for the teaching of mathematics (ME, 2007; NCTM, 2000; Stein et al., 2008). Furthermore, his misunderstanding of the situation described in the episode, in mathematical terms, may have hindered him from making adequate decisions regarding how to possibly continue the episode, i.e., from showing a deeper instructional knowledge. Indeed, as we point out next, a poor knowledge of mathematics seems to be associated to a weakened instructional knowledge.

Though having a short teaching experience, Joana is not a certified mathematics teacher, thus being enrolled in the same program as Júlio and Carlos. Joana's knowledge of mathematics seems to exhibit several weaknesses. For example, she states that

> Rita's conjecture was proved and it was incorrect, since the students checked for a large array of numbers, even bigger than 100, thus establishing a degree of certainty in their answers and even finding numbers like, for example '21' which though ending in 1 is not prime since 3 divides 21.

In fact, Joana does not seem to understand the meaning of conjecture nor what it takes to prove (or refute) a conjecture. Though acknowledging the value of the testing procedure, she completely mixes the two reciprocal implications involved in the episode.

Since Joana does not identify Rita's conjecture nor its reciprocal, it is not surprising that her suggestions for continuing the episode make little sense:

> The teacher should have let the students reach the conclusion that 'all numbers ending in 1, 3, 7, 9' are not prime and she should not have written on the board and telling the conclusion. Maybe saying the students should conclude or even writing only the sentence 'all numbers ending in 1, 3, 7, or 9 are prime; do you agree?' because, by saying 'So, see if it is true' she is implicitly telling the students that something is wrong.

Joana does not understand the teacher’s intentions when writing on the board the two implications, seeing the teacher’s actions as offering the students a clue for what would be correct or incorrect.

Both Joana and Carlos show weaknesses in their mathematical knowledge; however, Joana seems to have more problems in this regard than her colleague. Having completed a bachelor degree several years ago and not holding a major in mathematics, Joana has a different academic background from her colleagues’, who majored in mathematics the year before enrolling in the master’s degree. One could suggest that such differences in background could account for the differences in mathematical knowledge evidenced in the prospective teachers’ analyses of the episode. However, Carlos, whose background and grade point average is similar to Júlio’s, also shows gaps in his mathematical knowledge. In addition, Joana, unlike her colleagues, did have some teaching experience (besides other experiences in the job marked); nevertheless, her knowledge of instructional processes in the classroom emerges as much weaker than that of Júlio or Carlos.

Raising Issues for Teacher Education

The examples provided support the claim that, when one does not comprehend a mathematical situation, one can hardly make adequate instructional decisions (Kahan, Cooper, & Bethea, 2003). In particular, the orchestration of productive mathematical discussions and the systematization of (new) knowledge, two complex communicative actions, essential aspects of the teacher’s role within the current Portuguese curricular orientations, cannot be adequately approached if the teacher’s knowledge of the mathematics underneath the teaching situation is not sound (Martinho & Ponte, 2009; Ponte, in press; Tomás Ferreira et al., 2011). In other words, the teacher cannot make adequate instructional decisions (showing appropriate knowledge of classroom instructional processes) if s/he does not hold a vast, solid, and interrelated knowledge of mathematics.

The analysis of a classroom episode in a teacher education context as the one described above allows that prospective teachers understand the situations reported in the episode and search for information as many times as needed, since the episode can be read repeatedly. Very shortly, in their student teaching experience, these prospective teachers will be asked to do something much more demanding: in a classroom context, they will have to immediately comprehend what is at stake and weigh the various options to make adequate instructional decisions. The challenges involved are immense and we believe that the work around classroom episodes, focused on aspects of the teacher’s didactical knowledge, will help these future teachers to face those challenges. However, we question ourselves about how the current structure of teacher education programs is helping prospective teachers developing a sound mathematical knowledge that may support the development of a strong didactical knowledge. Since all prospective teachers had a heavy load of mathematics courses in their bachelor degree (in which various techniques of mathematical proof were addressed), we would not expect so many difficulties in understanding basic notions, such as conjectures, examples and counter-examples, nor so many obstacles in comprehending a teaching situation, such as the one described in the episode. The data suggests prospective teachers do not develop adequate conceptions about mathematical proof (in their bachelor degree) which hinders the development of their didactical knowledge (in particular the dimension of instructional knowledge) when enrolled in a teacher certification program. As mathematics educators, how can we deal with this reality? What connections may be made with mathematics instructors so that basic mathematical notions do not constitute a barrier to developing instructional knowledge and taking the most of a teacher education program? What happens in other countries with similar higher education configurations?

References

Bishop, A. J.,, & Goffree, F. (1986). Classroom organization and dynamics. In B. Christiansen, A. G. Howson, & M. Otte (Eds.), *Perspectives on mathematics education* (pp. 309-365). Dordrecht: Reidel.

Boavida, A. (2001). Um olhar sobre o ensino da demonstração em Matemática. *Educação e Matemática, 63,* 11-15.

Empson, S. B., & Jacobs, V. R. (2008). Learning to listen to children's mathematical thinking. In D. Tirosh & T. Wood (Eds.), *The international handbook of mathematics teacher education* (Vol. 2, pp. 251-281). Rotherdam, the Netherlands: Sense Publishers.

Kahan, J. A., Cooper, D. A., & Bethea, K. A. (2003). The role of mathematics teachers' content knowledge in their teaching: A framework for research applied to a study of student teachers. *Journal of Mathematics Teacher Education, 6*(3), 223-252.

Martinho, M. H., & Ponte, J. P. (2009). Communication in the classroom: Practice and reflection of a mathematics teacher. *Quaderni di Ricerca in Didattica (Matematica)*, Supplemento n.2 al n. 19.

ME (2007). *Programa de matemática do ensino básico*. Lisboa, Portugal: DGIDC.

National Council of Teachers of Mathematics. (1991). *Professional standards for school mathematics*. Reston, VA. NCTM.

National Council of Teachers of Mathematics (2000). *Principles and standards for school mathematics*. Reston, VA: NCTM.

Ponte, J. P. (1999). Didácticas específicas e construção do conhecimento profissional. In J. Tavares, A. Pereira, A. Pedro, & H. Sá (Eds.), *Investigar e formar em educação – Proceedings of XI EIEM* (pp. 59-72). Porto, Portugal: SPCE.

Ponte, J. P. (in press). Estudando o conhecimento e o desenvolvimento profissional do professor de matemática. In N. Planas (Ed.), *Educación matemática: Teoría, critica y prática*. Barcelona: Graó.

Ponte, J. P., & Oliveira, H. (2002). Remar contra a maré: A construção do conhecimento e da identidade profissional na formação inicial. *Revista de Educação, 11*(2), 145-163.

Ruthven, K., Hofmann, R., & Mercer, N. (2011). A dialogic approach to plenary problem synthesis. *Proceedings of the 35th annual conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 81-88). Ankara, Turkey: PME.

Stein, M. K., & Smith. M. (1998). Mathematical tasks as a framework for reflection: From research to practice. *Mathematics Teaching in the Middle School*, *3*(4), 268-275.

Stein, M. K., Engle, R. A., Smith, M. S., & Hughes, E. K. (2008). Orchestrating productive mathematical discussions: Five practices for helping teachers move beyond show and tell. *Mathematical Thinking and Learning, 10*(4), 313-340.

Tomás Ferreira, R. A. (2005). *Portuguese student teachers' evolving teaching modes: A modified teacher development experiment.* Doctoral dissertation. Illinois State University, USA.

Tomás Ferreira, R. A., Martinho, M. H., Menezes, L. (2011). "Teachable Moments": Oportunidades a não perder. *Educação e Matemática, 115*, 39-45.